ON THE OSCILLATION OF THE SOLUTIONS TO LINEAR DIFFERENCE EQUATIONS WITH VARIABLE DELAY

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ABSTRACT. A new criterion for the oscillation of the solutions to linear difference equations with variable delay is established. This criterion is based on a new fundamental lemma, which provides a useful inequality for the nonoscillatory solutions of the delay difference equations considered.

1. INTRODUCTION

In the last two decades, the study of difference equations has attracted significant interest by many researchers. This is due, in a large part, to the rapidly increasing number of applications of the theory of difference equations to various fields of applied sciences and technology. In particular, the oscillation theory of difference equations has been extensively developed. See [1–27] and the references cited therein. The present paper deals with the oscillation of linear difference equations with variable delay.

Consider the delay difference equation

$$\Delta x(n) + p(n)x(\tau(n)) = 0, \tag{1.1}$$

where $(p(n))_{n\geq 0}$ *is a sequence of nonnegative real numbers, and* $(\tau(n))_{n\geq 0}$ *is a sequence of integers such that*

$$\tau(n) \leq n-1 \ \textit{ for all } n \geq 0, \qquad \textit{and} \qquad \lim_{n\to\infty} \tau(n) = \infty.$$

Here, Δ stands for the usual forward difference operator defined by

$$\Delta h(n) = h(n+1) - h(n), \quad n \geq 0,$$

for any sequence of real numbers $(h(n))_{n\geq 0}$.

Set

$$k = -\min_{n\geq 0} \tau(n).$$

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(Clearly, k is a positive integer.)

By a *solution* of the delay difference equation (1.1), we mean a sequence of real numbers $(x(n))_{n\geq -k}$ which satisfies (1.1) for all $n \geq 0$. It is clear that, for each choice of real numbers c_{-k}, $c_{-k+1}, \ldots$, c_{-1}, c_0, there exists a unique solution $(x(n))_{n\geq -k}$ of (1.1) which satisfies the initial conditions $x(-k) = c_{-k}$, $x(-k+1) = c_{-k+1}, \ldots$, $x(-1) = c_{-1}$, $x(0) = c_0$.

As usual, a solution $(x(n))_{n\geq -k}$ of the delay difference equation (1.1) is called *oscillatory* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative, and otherwise the solution is said to be *nonoscillatory*.

In the special case where the delay $(n - \tau(n))_{n\geq 0}$ is a constant, the delay difference equation (1.1) becomes

$$\Delta x(n) + p(n)x(n-k) = 0, \tag{1.2}$$

where *k is a positive integer*.

In 1989, Erbe and Zhang [8] established that all solutions of (1.2) are oscillatory if

$$\liminf_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} \tag{1.3}$$

or

$$\limsup_{n\to\infty} \sum_{j=n-k}^{n} p(j) > 1. \tag{1.4}$$

In the same year, 1989, Ladas, Philos and Sficas [13] proved that a sufficient condition for all solutions of (1.2) to be oscillatory is that

$$\liminf_{n\to\infty} \left[\frac{1}{k}\sum_{j=n-k}^{n-1} p(j)\right] > \frac{k^k}{(k+1)^{k+1}}. \tag{1.5}$$

(Clearly, condition (1.5) improves (1.3).) A substantial improvement of this oscillation criterion has been presented, in 2004, by Philos, Purnaras and Stavroulakis [19].

Since 1989, a large number of related papers have been published. See [2–7, 10–12, 14–27] and the references cited therein. Most of these papers are concerning the special case of the delay difference equation (1.2), while a small number of these papers are dealing with the general case of the delay difference equation (1.1), in which the delay $(n - \tau(n))_{n\geq 0}$ is variable.

It is interesting to establish sufficient oscillation conditions for the equation (1.2), in the case where neither (1.4) not (1.5) is satisfied. This question has been investigated by several authors. See, for example, Chatzarakis and Stavroulakis [3] and the references cited therein.

Under the hypothesis that the sequence $(\tau(n))_{n\geq 0}$ is increasing, from Chatzarakis, Koplatadze and Stavroulakis [2], it follows that all solutions of (1.1) are oscillatory if

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1. \tag{1.6}$$

This result generalizes the oscillation criterion (1.4). In 1991, Philos [16] extended the oscillation criterion (1.5) to the general case of the delay difference equation (1.1). More precisely, it has been established in [16] that, if the sequence $(\tau(n))_{n\geq 0}$ is increasing, then the condition

$$\liminf_{n\to\infty} \left[\frac{1}{n-\tau(n)} \sum_{j=\tau(n)}^{n-1} p(j) \right] > \limsup_{n\to\infty} \frac{(n-\tau(n))^{n-\tau(n)}}{(n-\tau(n)+1)^{n-\tau(n)+1}} \tag{1.7}$$

suffices for the oscillation of all solutions of (1.1).

As it has been mentioned above, it is an interesting problem to find new sufficient conditions for the oscillation of all solutions of the delay difference equation (1.1), in the case where neither (1.6) nor (1.7) is satisfied. Very recently, Chatzarakis, Koplatadze and Stavroulakis [2] investigated for the first time this question for the delay difference equation (1.1) in the case of a general delay argument $\tau(n)$ and derived a lemma and a theorem which, in the special case where the sequence $(\tau(n))_{n\geq 0}$ is increasing, can be formulated as follows:

Lemma 1.1 ([2]). *Assume that the sequence $(\tau(n))_{n\geq 0}$ is increasing, and set*

$$\alpha = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j). \tag{1.8}$$

Let $(x(n))_{n\geq -k}$ be a nonoscillatory solution of the delay difference equation (1.1). Then we have:

(i) *If $0 < \alpha \leq 1$, then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq (1-\sqrt{1-\alpha})^2. \tag{1.9}$$

(ii) *If $0 < \alpha < 1$ and, in addition,*

$$p(n) \geq 1-\sqrt{1-\alpha} \quad \textit{for all large } n, \tag{1.10}$$

then

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq \alpha \frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}}. \tag{1.11}$$

Theorem 1.1 ([2]). *Assume that the sequence $(\tau(n))_{n\geq 0}$ is increasing, and define α by (1.8). Then we have:*

(I) *If $0 < \alpha \leq 1$, then the condition*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \left(1-\sqrt{1-\alpha}\right)^2 \tag{1.12}$$

is sufficient for all solutions of the delay difference equation (1.1) to be oscillatory.

(II) *If $0 < \alpha < 1$ and, in addition, (1.10) holds, then the condition*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \alpha \frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}} \tag{1.13}$$

is sufficient for all solutions of (1.1) to be oscillatory.

In this paper, new oscillation criteria for the solutions of (1.1) are established, which substantially improve the corresponding criteria in [2], as well as all the known corresponding criteria concerning the special case of the equation (1.2).

Our main result will be stated and proved in Section 3. Section 2 is devoted to establishing a basic lemma, which plays a crucial role in proving our main result.

2. A BASIC LEMMA

The proof of our main result (i.e., of Theorem 3.1 given in the next section) is essentially based on the following lemma.

Lemma 2.1. *Assume that the sequence $(\tau(n))_{n\geq 0}$ is increasing, and set*

$$\alpha = \liminf_{n\to\infty} \sum_{j=\tau(n)}^{n-1} p(j). \tag{2.1}$$

Let $(x(n))_{n\geq -k}$ be a nonoscillatory solution of the delay difference equation (1.1). Then we have:

(i) *If* $0 < \alpha \le \frac{1}{2}$, *then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \ge \frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right). \tag{2.2}$$

(ii) *If* $0 < \alpha \le 6-4\sqrt{2}$ *and, in addition,*

$$p(n) \ge \frac{\alpha}{2} \quad \textit{for all large } n, \tag{2.3}$$

then

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \ge \frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right). \tag{2.4}$$

Note. If $0 < \alpha \le \frac{1}{2}$, then $1-2\alpha \ge 0$ and

$$0 < \frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right) < \frac{1}{2}.$$

Also, when $0 < \alpha \le 6-4\sqrt{2}$ (clearly, $6-4\sqrt{2} < \frac{1}{2}$), we have $4-12\alpha+\alpha^2 \ge 0$ and

$$0 < \frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right) < \frac{1}{2}.$$

Moreover, provided that $0 < \alpha \le 6-4\sqrt{2}$, we also have

$$\frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right) > \frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right). \tag{2.5}$$

Therefore, in the case where $0 < \alpha \le 6-4\sqrt{2}$ and (2.3) holds, inequality (2.5) guarantees that (2.4) is an improvement of (2.2).

Proof of Lemma 2.1. Since the solution $(x(n))_{n\ge -k}$ of the delay difference equation (1.1) is nonoscillatory, it is either eventually positive or eventually negative. As $(-x(n))_{n\ge -k}$ is also a solution of (1.1), we may (and do) restrict ourselves only to the case where $x(n) > 0$ for all large n. Let $\rho \ge -k$ be an integer such that $x(n) > 0$ for all $n \ge \rho$, and consider an integer $r \ge 0$ so that $\tau(n) \ge \rho$ for $n \ge r$ (clearly, $r > \rho$). Then it follows immediately from (1.1) that $\Delta x(n) \le 0$ for every $n \ge r$, which means that the sequence $(x(n))_{n\ge r}$ is decreasing.

Assume that $0 < \alpha \le \frac{1}{2}$, where α is defined by (2.1). Consider an arbitrary real number ϵ with $0 < \epsilon < \alpha$. Then we can choose an integer $n_0 > r$ such that $\tau(n) \ge r$ for $n \ge n_0$, and

$$\sum_{j=\tau(n)}^{n-1} p(j) \ge \alpha - \epsilon \quad \text{for all } n \ge n_0. \tag{2.6}$$

Furthermore, let us consider an arbitrary real number ω with $0 < \omega < \alpha - \epsilon$. We will establish the following claim.

Claim. *For each $n \geq n_0$, there exists an integer $n^* \geq n$ such that $\tau(n^*) \leq n - 1$, and*

$$\sum_{j=n}^{n^*} p(j) \geq \omega \tag{2.7}$$

and

$$\sum_{j=\tau(n^*)}^{n-1} p(j) > (\alpha - \epsilon) - \omega. \tag{2.8}$$

To prove this claim, let us consider an arbitrary integer $n \geq n_0$. Assume, first, that $p(n) \geq \omega$, and choose $n^* = n$. Then $\tau(n^*) = \tau(n) \leq n - 1$. Moreover, we have

$$\sum_{j=n}^{n^*} p(j) = \sum_{j=n}^{n} p(j) = p(n) \geq \omega$$

and, by (2.6),

$$\sum_{j=\tau(n^*)}^{n-1} p(j) = \sum_{j=\tau(n)}^{n-1} p(j) \geq a - \epsilon > (a - \epsilon) - \omega.$$

So, (2.7) and (2.8) are fulfilled. Next, we suppose that $p(n) < \omega$. It is not difficult to see that (2.6) guarantees that

$$\sum_{j=0}^{\infty} p(j) = \infty.$$

In particular, it holds

$$\sum_{j=n}^{\infty} p(j) = \infty.$$

Thus, as $p(n) < \omega$, there always exists an integer $n^* > n$ so that

$$\sum_{j=n}^{n^*-1} p(j) < \omega \tag{2.9}$$

and (2.7) holds. We assert that $\tau(n^*) \leq n - 1$. Otherwise, $\tau(n^*) \geq n$. We also have $\tau(n^*) \leq n^* - 1$. Hence, in view of (2.9), we get

$$\sum_{j=\tau(n^*)}^{n^*-1} p(j) \leq \sum_{j=n}^{n^*-1} p(j) < \omega.$$

On the other hand, (2.6) gives

$$\sum_{j=\tau(n^*)}^{n^*-1} p(j) \geq \alpha - \epsilon > \omega.$$

We have arrived at a contradiction, which shows our assertion. Furthermore, by using (2.6) (for the integer n^*) as well as (2.9), we obtain

$$\sum_{j=\tau(n^*)}^{n-1} p(j) = \sum_{j=\tau(n^*)}^{n^*-1} p(j) - \sum_{j=n}^{n^*-1} p(j) > (\alpha - \epsilon) - \omega$$

and consequently (2.8) holds true. Our claim has been proved.

Next, we choose an integer $N > n_0$ such that $\tau(n) \geq n_0$ for $n \geq N$. Let us consider an arbitrary integer $n \geq N$. By our claim, there exists an integer $n^* \geq n$ such that $\tau(n^*) \leq n-1$, and (2.7) and (2.8) hold. By taking into account the facts that the sequence $(\tau(s))_{s\geq 0}$ is increasing and that the sequence $(x(t))_{t\geq r}$ is decreasing and by using (2.7), from (1.1), we obtain

$$x(n) - x(n^*+1) = \sum_{j=n}^{n^*} p(j)x(\tau(j)) \geq \left[\sum_{j=n}^{n^*} p(j)\right] x(\tau(n^*)) \geq \omega x(\tau(n^*))$$

and consequently

$$x(n) \geq x(n^*+1) + \omega x(\tau(n^*)). \tag{2.10}$$

Furthermore, by taking again into account the facts that $(\tau(s))_{s\geq 0}$ is increasing and that $(x(t))_{t\geq r}$ is decreasing and by using (2.8), from (1.1), we derive

$$\begin{aligned} x(\tau(n^*)) - x(n) &= \sum_{j=\tau(n^*)}^{n-1} p(j)x(\tau(j)) \geq \left[\sum_{j=\tau(n^*)}^{n-1} p(j)\right] x(\tau(n-1)) \\ &> [(\alpha-\epsilon)-\omega]\, x(\tau(n-1)) \end{aligned}$$

and so

$$x(\tau(n^*)) > x(n) + [(\alpha-\epsilon)-\omega]x(\tau(n-1)). \tag{2.11}$$

By (2.10) and (2.11), we get

$$x(n) \geq x(n^*+1) + \omega x(\tau(n^*)) > \omega x(\tau(n^*)) > \omega \{x(n) + [(\alpha-\epsilon)-\omega]x(\tau(n-1))\}$$

and hence

$$x(n) > \omega \frac{(\alpha-\epsilon)-\omega}{1-\omega} x(\tau(n-1)).$$

We have thus proved that

$$x(n) > \omega\lambda_1 x(\tau(n-1)) \quad \text{for all } n \geq N, \tag{2.12}$$

where

$$\lambda_1 = \frac{(\alpha - \epsilon) - \omega}{1 - \omega}.$$

Now, let n be an arbitrary integer with $n \geq N$. By using our claim, we conclude that there exists an integer $n^* \geq n$ such that $\tau(n^*) \leq n-1$, and (2.7) and (2.8) are satisfied. Then (2.10) and (2.11) are also fulfilled. Moreover, in view of (2.12) (for the integer $n^* + 1$), we have

$$x(n^* + 1) > \omega\lambda_1 x(\tau(n^*)). \tag{2.13}$$

By the use of (2.10), (2.13) and (2.11), we obtain

$$\begin{aligned} x(n) &\geq x(n^* + 1) + \omega x(\tau(n^*)) > \omega\lambda_1 x(\tau(n^*)) + \omega x(\tau(n^*)) \\ &= \omega(\lambda_1 + 1)x(\tau(n^*)) > \omega(\lambda_1 + 1)\left\{x(n) + [(\alpha - \epsilon) - \omega]x(\tau(n-1))\right\}, \end{aligned}$$

which gives

$$[1 - \omega(\lambda_1 + 1)]x(n) > \omega(\lambda_1 + 1)[(\alpha - \epsilon) - \omega]x(\tau(n-1)).$$

This implies, in particular, that

$$1 - \omega(\lambda_1 + 1) > 0.$$

Consequently,

$$x(n) > \omega\frac{(\lambda_1 + 1)[(\alpha - \epsilon) - \omega]}{1 - \omega(\lambda_1 + 1)}x(\tau(n-1)).$$

Thus, it has been shown that

$$x(n) > \omega\lambda_2 x(\tau(n-1)) \quad \text{for all } n \geq N,$$

where

$$\lambda_2 = \frac{(\lambda_1 + 1)[(\alpha - \epsilon) - \omega]}{1 - \omega(\lambda_1 + 1)}.$$

Following the above procedure, we can inductively construct a sequence of positive real numbers $(\lambda_\nu)_{\nu \geq 1}$ with

$$1 - \omega(\lambda_\nu + 1) > 0 \quad (\nu = 1, 2, ...)$$

and

$$\lambda_{\nu+1} = \frac{(\lambda_\nu + 1)[(\alpha - \epsilon) - \omega]}{1 - \omega(\lambda_\nu + 1)} \quad (\nu = 1, 2, ...)$$

such that

$$x(n) > \omega\lambda_\nu x(\tau(n-1)) \quad \text{for all } n \geq N \quad (\nu = 1, 2, ...). \tag{2.14}$$

As $\lambda_1 > 0$, we obtain

$$\lambda_2 = \frac{(\lambda_1 + 1)[(\alpha - \epsilon) - \omega]}{1 - \omega(\lambda_1 + 1)} > \frac{(\alpha - \epsilon) - \omega}{1 - \omega} = \lambda_1,$$

i.e., $\lambda_2 > \lambda_1$. By an easy induction, one can see that the sequence $(\lambda_\nu)_{\nu\geq 1}$ is strictly increasing. Furthermore, by taking into account the fact that the sequence $(x(t))_{t\geq r}$ is decreasing and by using (2.14) (for $n = N$), we get

$$x(\tau(N-1)) \geq x(N) > \omega\lambda_\nu x(\tau(N-1)) \quad (\nu = 1, 2, ...).$$

Therefore, for each $\nu \geq 1$, we have $\omega\lambda_\nu < 1$, i.e., $\lambda_\nu < \frac{1}{\omega}$. This ensures that the sequence $(\lambda_\nu)_{\nu\geq 1}$ is bounded. Since $(\lambda_\nu)_{\nu\geq 1}$ is a strictly increasing and bounded sequence of positive real numbers, it follows that $\lim_{\nu\to\infty} \lambda_\nu$ exists as a positive real number. Set

$$\Lambda = \lim_{\nu\to\infty} \lambda_\nu.$$

Then (2.14) gives

$$x(n) \geq \omega\Lambda x(\tau(n-1)) \quad \text{for all } n \geq N. \tag{2.15}$$

By the definition of $(\lambda_\nu)_{\nu\geq 1}$, we have

$$\Lambda = \frac{(\Lambda+1)[(\alpha-\epsilon)-\omega]}{1-\omega(\Lambda+1)},$$

i.e.,

$$\omega\Lambda^2 - [1-(\alpha-\epsilon)]\Lambda + [(\alpha-\epsilon)-\omega] = 0.$$

Hence, either

$$\Lambda = \frac{1}{2\omega}\left\{1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)+[(\alpha-\epsilon)-2\omega]^2}\right\}$$

or

$$\Lambda = \frac{1}{2\omega}\left\{1-(\alpha-\epsilon)+\sqrt{1-2(\alpha-\epsilon)+[(\alpha-\epsilon)-2\omega]^2}\right\}.$$

In both cases, it holds

$$\Lambda \geq \frac{1}{2\omega}\left\{1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)+[(\alpha-\epsilon)-2\omega]^2}\right\}.$$

Thus, from (2.15), it follows that

$$x(n) \geq \frac{1}{2}\left\{1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)+[(\alpha-\epsilon)-2\omega]^2}\right\} x(\tau(n-1)) \quad \text{for all } n \geq N. \tag{2.16}$$

But, we can immediately see that the function

$$f(\omega) = \frac{1}{2}\left\{1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)+[(\alpha-\epsilon)-2\omega]^2}\right\} \quad \text{for } 0 < \omega < \alpha-\epsilon$$

attains its maximum at the point $\omega = \frac{\alpha-\epsilon}{2}$. So, by choosing $\omega = \frac{\alpha-\epsilon}{2}$, from (2.16) we obtain
(2.17)

$$x(n) \geq \frac{1}{2}\left[1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)}\right] x(\tau(n-1)) \quad \text{for all } n \geq N.$$

Inequality (2.17) gives

$$x(n+1) \geq \frac{1}{2}\left[1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)}\right] x(\tau(n)) \quad \text{for every } n \geq N-1$$

or

$$\frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{2}\left[1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)}\right] \quad \text{for all } n \geq N-1.$$

Consequently,

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{2}\left[1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)}\right].$$

The last inequality holds true for all real numbers ϵ with $0 < \epsilon < \alpha$. Hence, we can obtain (2.2). The proof of Part (i) of the lemma has been completed.

In the remainder of the proof of the lemma, it will be assumed that $0 < \alpha \leq 6 - 4\sqrt{2}$ (which implies that $0 < \alpha < \frac{1}{2}$) and, in addition, that (2.3) holds. Because of (2.3), we can consider an integer $L \geq N$ such that $p(n) \geq \frac{\alpha}{2}$ for every $n \geq L$. Then

$$p(n) > \frac{\alpha-\epsilon}{2} \quad \text{for all } n \geq L. \tag{2.18}$$

By (2.17), we have

$$x(n) \geq \theta_1 x(\tau(n-1)) \quad \text{for all } n \geq L, \tag{2.19}$$

where

$$\theta_1 = \frac{1}{2}\left[1-(\alpha-\epsilon)-\sqrt{1-2(\alpha-\epsilon)}\right].$$

Let us consider an arbitrary integer $n \geq L$. By using (2.18) as well as (2.19) (for the integer $n+1$), from (1.1) we obtain

$$x(n) = x(n+1)+p(n)x(\tau(n)) > x(n+1)+\frac{\alpha-\epsilon}{2}x(\tau(n)) \geq \theta_1 x(\tau(n))+\frac{\alpha-\epsilon}{2}x(\tau(n))$$

and consequently

$$x(n) > \left(\theta_1 + \frac{\alpha-\epsilon}{2}\right) x(\tau(n)). \tag{2.20}$$

Furthermore, by taking into account the facts that $(\tau(s))_{s\geq 0}$ is increasing and that $(x(t))_{t\geq r}$ is decreasing and by using (2.6), from (1.1), we derive

$$x(\tau(n))-x(n)=\sum_{j=\tau(n)}^{n-1}p(j)x(\tau(j))\geq\left[\sum_{j=\tau(n)}^{n-1}p(j)\right]x(\tau(n-1))\geq(\alpha-\epsilon)x(\tau(n-1))$$

and hence

$$x(\tau(n))\geq x(n)+(\alpha-\epsilon)x(\tau(n-1)). \tag{2.21}$$

A combination of (2.20) and (2.21) gives

$$x(n)>\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)[x(n)+(\alpha-\epsilon)x(\tau(n-1))],$$

i.e.,

$$\left[1-\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)\right]x(n)>\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)(\alpha-\epsilon)x(\tau(n-1)).$$

This guarantees, in particular, that

$$1-\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)>0.$$

So,

$$x(n)>\frac{\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)(\alpha-\epsilon)}{1-\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)}x(\tau(n-1)).$$

We have thus proved that

$$x(n)>\theta_2x(\tau(n-1))\quad\text{for all } n\geq L,$$

where

$$\theta_2=\frac{\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)(\alpha-\epsilon)}{1-\left(\theta_1+\frac{\alpha-\epsilon}{2}\right)}.$$

By the arguments applied previously, a sequence of positive real numbers $(\theta_\nu)_{\nu\geq 1}$ can inductively constructed, which satisfies

$$1-\left(\theta_\nu+\frac{\alpha-\epsilon}{2}\right)>0\quad(\nu=1,2,...)$$

and

$$\theta_{\nu+1}=\frac{\left(\theta_\nu+\frac{\alpha-\epsilon}{2}\right)(\alpha-\epsilon)}{1-\left(\theta_\nu+\frac{\alpha-\epsilon}{2}\right)}\quad(\nu=1,2,...);$$

this sequence is such that (2.19) holds, and

$$x(n)>\theta_\nu x(\tau(n-1))\quad\text{for all } n\geq L\qquad(\nu=2,3,...). \tag{2.22}$$

By the use of the definitions of θ_1 and θ_2, it is a matter of elementary calculations to find

$$\theta_2 = 1 - (\alpha - \epsilon) - \sqrt{1 - 2(\alpha - \epsilon)}, \quad \text{i.e.,} \quad \theta_2 = 2\theta_1.$$

So, $\theta_2 > \theta_1$. By induction, we can easily verify that the sequence $(\theta_\nu)_{\nu \geq 1}$ is strictly increasing. Furthermore, by taking into account the fact that $(x(t))_{t \geq r}$ is decreasing and by using (for $n = L$) inequality (2.22), we obtain

$$x(\tau(L-1)) \geq x(L) > \theta_\nu x(\tau(L-1)) \quad (\nu = 2, 3, ...).$$

Hence, $\theta_\nu < 1$ for every $\nu \geq 2$, which guarantees the boundedness of the sequence $(\theta_\nu)_{\nu \geq 1}$. Thus, $\lim_{\nu \to \infty} \theta_\nu$ exists as a positive real number. Define

$$\Theta = \lim_{\nu \to \infty} \theta_\nu.$$

Then it follows from (2.22) that

$$x(n) \geq \Theta x(\tau(n-1)) \quad \text{for all } n \geq L. \tag{2.23}$$

In view of the definition of $(\theta_\nu)_{\nu \geq 1}$, the number Θ satisfies

$$\Theta = \frac{\left(\Theta + \frac{\alpha - \epsilon}{2}\right)(\alpha - \epsilon)}{1 - \left(\Theta + \frac{\alpha - \epsilon}{2}\right)}$$

or, equivalently,

$$2\Theta^2 - [2 - 3(\alpha - \epsilon)]\Theta + (\alpha - \epsilon)^2 = 0.$$

So, either

$$\Theta = \frac{1}{4}\left[2 - 3(\alpha - \epsilon) - \sqrt{4 - 12(\alpha - \epsilon) + (\alpha - \epsilon)^2}\right]$$

or

$$\Theta = \frac{1}{4}\left[2 - 3(\alpha - \epsilon) + \sqrt{4 - 12(\alpha - \epsilon) + (\alpha - \epsilon)^2}\right].$$

Note that, because of $0 < \alpha - \epsilon < 6 - 4\sqrt{2}$, it holds

$$4 - 12(\alpha - \epsilon) + (\alpha - \epsilon)^2 > 0.$$

We always have

$$\Theta \geq \frac{1}{4}\left[2 - 3(\alpha - \epsilon) - \sqrt{4 - 12(\alpha - \epsilon) + (\alpha - \epsilon)^2}\right]$$

and consequently (2.23) gives

$$x(n) \geq \frac{1}{4}\left[2 - 3(\alpha - \epsilon) - \sqrt{4 - 12(\alpha - \epsilon) + (\alpha - \epsilon)^2}\right] x(\tau(n-1)) \quad \text{for all } n \geq L.$$

Finally, we see that the last inequality can equivalently be written as follows

$$x(n+1) \geq \frac{1}{4}\left[2-3(\alpha-\epsilon)-\sqrt{4-12(\alpha-\epsilon)+(\alpha-\epsilon)^2}\right]x(\tau(n)) \quad \text{for } n \geq L-1,$$

i.e.,

$$\frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{4}\left[2-3(\alpha-\epsilon)-\sqrt{4-12(\alpha-\epsilon)+(\alpha-\epsilon)^2}\right] \quad \text{for all } n \geq L-1.$$

Therefore,

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{4}\left[2-3(\alpha-\epsilon)-\sqrt{4-12(\alpha-\epsilon)+(\alpha-\epsilon)^2}\right].$$

As this inequality is satisfied for all real numbers ϵ with $0 < \epsilon < \alpha$, we can conclude that (2.4) holds true. So, Part (ii) of the lemma has been proved.

The proof of the lemma is complete.

Remark 2.1. Observe the following:

(i) When $0 < \alpha \leq \frac{1}{2}$, it is easy to verify that

$$\frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right) > \left(1-\sqrt{1-\alpha}\right)^2,$$

and therefore inequality (2.2) improves inequality (1.9).

(ii) When $0 < \alpha \leq 6-4\sqrt{2}$, because

$$1-\sqrt{1-\alpha} > \frac{\alpha}{2},$$

we see that assumption (2.3) is weaker than assumption (1.10), and, moreover, we can show that

$$\frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right) > \alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}}$$

and so inequality (2.4) is an improvement of inequality (1.11).

Remark 2.2. It is an open question whether inequality (2.2) can be improved as follows

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(\tau(n))} \geq \frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha-\alpha^2}\right), \tag{2.24}$$

provided that $0 < \alpha \leq -1 + \sqrt{2}$. This question arises from a lemma due to Chen and Yu [4]; according to it, if $0 < \alpha_0 \leq (\frac{k}{k+1})^{k+1}$, where

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-k}^{n-1} p(j),$$

then every nonoscillatory solution of the delay difference equation (1.2) satisfies

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(n-k)} \geq \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0 - \alpha_0^2}\right).$$

Observe, however, that when $0 < \alpha \leq 6 - 4\sqrt{2}$, it is easy to show that

$$\frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) > \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right),$$

and therefore, in this case and when (2.3) holds, inequality (2.4) in Lemma 2.1 improves the above inequality (2.24).

3. THE MAIN RESULT

Our main result is the following theorem.

Theorem 3.1. *Assume that the sequence* $(\tau(n))_{n\geq 0}$ *is increasing, and define* α *by* (2.1). *Then we have:*

(I) *If* $0 < \alpha \leq \frac{1}{2}$, *then the condition*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha}\right) \tag{3.1}$$

is sufficient for all solutions of the delay difference equation (1.1) *to be oscillatory.*

(II) *If* $0 < \alpha \leq 6 - 4\sqrt{2}$ *and, in addition,* (2.3) *holds, then the condition*

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{4}\left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2}\right) \tag{3.2}$$

is sufficient for all solutions of (1.1) *to be oscillatory.*

Proof. Suppoce, for the sake of contradiction, that the delay difference equation (1.1) admits a nonoscillatory solution $(x(n))_{n\geq -k}$. Since $(-x(n))_{n\geq -k}$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $(x(n))_{n\geq -k}$ is eventually positive.

Consider an integer $\rho \geq -k$ so that $x(n) > 0$ for every $n \geq \rho$, and let $r \geq 0$ be an integer such that $\tau(n) \geq \rho$ for $n \geq r$ (clearly, $r > \rho$). Then from (1.1) we immediately obtain $\Delta x(n) \leq 0$ for all $n \geq r$, and consequently the sequence $(x(n))_{n \geq r}$ is decreasing.

Now, we consider an integer $n_0 > r$ such that $\tau(n) \geq r$ for $n \geq n_0$. Furthermore, we choose an integer $N > n_0$ so that $\tau(n) \geq n_0$ for $n \geq N$. Then, by taking into account the facts that the sequence $(\tau(s))_{s \geq 0}$ is increasing and that the sequence $(x(t))_{t \geq r}$ is decreasing, from (1.1) we obtain, for every $n \geq N$,

$$x(\tau(n)) - x(n+1) = \sum_{j=\tau(n)}^{n} p(j) x(\tau(j)) \geq \left[\sum_{j=\tau(n)}^{n} p(j) \right] x(\tau(n)).$$

Consequently,

$$\sum_{j=\tau(n)}^{n} p(j) \leq 1 - \frac{x(n+1)}{x(\tau(n))} \quad \text{for all } n \geq N,$$

which gives

$$\limsup_{n \to \infty} \sum_{j=\tau(n)}^{n} p(j) \leq 1 - \liminf_{n \to \infty} \frac{x(n+1)}{x(\tau(n))}. \tag{3.3}$$

Assume, first, that $0 < \alpha \leq \frac{1}{2}$. Then, by Lemma 2.1, inequality (2.2) is fulfilled, and so (3.3) leads to

$$\limsup_{n \to \infty} \sum_{j=\tau(n)}^{n} p(j) \leq 1 - \frac{1}{2} \left(1 - \alpha - \sqrt{1 - 2\alpha} \right),$$

which contradicts condition (3.1).

Next, let us suppose that $0 < \alpha \leq 6 - 4\sqrt{2}$ and that (2.3) holds. Then Lemma 2.1 ensures that (2.4) is satisfied. Thus, from (3.3), it follows that

$$\limsup_{n \to \infty} \sum_{j=\tau(n)}^{n} p(j) \leq 1 - \frac{1}{4} \left(2 - 3\alpha - \sqrt{4 - 12\alpha + \alpha^2} \right),$$

which contradicts condition (3.2).

The proof of the theorem is complete.

As it has already been mentioned, Theorem 1.1 is presented in [2] in a more general form. More precisely, it is not assumed that the sequence $(\tau(n))_{n\geq 0}$ is increasing, but conditions (1.12) and (1.13) are replaced by the conditions

$$\limsup_{n\to\infty}\sum_{j=\sigma(n)}^{n} p(j) > 1-\left(1-\sqrt{1-\alpha}\right)^2$$

and

$$\limsup_{n\to\infty}\sum_{j=\sigma(n)}^{n} p(j) > 1-\alpha\frac{1-\sqrt{1-\alpha}}{\sqrt{1-\alpha}},$$

respectively, where the sequence of integers $(\sigma(n))_{n\geq 0}$ is defined by

$$\sigma(n) = \max_{0\leq s\leq n}\tau(s) \quad \text{for } n\geq 0. \tag{3.4}$$

Clearly, the sequence $(\sigma(n))_{n\geq 0}$ is increasing. Moreover, as it has been shown in [2], it holds

$$\liminf_{n\to\infty}\sum_{j=\sigma(n)}^{n-1} p(j) = \liminf_{n\to\infty}\sum_{j=\tau(n)}^{n-1} p(j). \tag{3.5}$$

Following [2], one can use (3.5) and apply Lemma 2.1 in [2] (cf. Philos [16] and Kordonis and Philos [10]) to establish the following generalization of Theorem 3.1.

Theorem 3.1$'$. *Let the sequence $(\sigma(n))_{n\geq 0}$ be defined by* (3.4), *and define α by* (2.1). *Then we have:*

(I)$'$ *If $0<\alpha\leq\frac{1}{2}$, then the condition*

$$\limsup_{n\to\infty}\sum_{j=\sigma(n)}^{n} p(j) > 1-\frac{1}{2}\left(1-\alpha-\sqrt{1-2\alpha}\right)$$

is sufficient for all solutions of the delay difference equation (1.1) *to be oscillatory.*

(II)$'$ *If $0<\alpha\leq 6-4\sqrt{2}$ and, in addition,* (2.3) *holds, then the condition*

$$\limsup_{n\to\infty}\sum_{j=\sigma(n)}^{n} p(j) > 1-\frac{1}{4}\left(2-3\alpha-\sqrt{4-12\alpha+\alpha^2}\right)$$

is sufficient for all solutions of (1.1) *to be oscillatory.*

Remark 3.1. Observe the following (cf. Remark 2.1):

(i) When $0<\alpha\leq\frac{1}{2}$, the condition (3.1) is weaker than the condition (1.12).

(ii) When $0 < \alpha \leq 6 - 4\sqrt{2}$, the conditions (2.3) and (3.2) are weaker than the conditions (1.10) and (1.13), respectively.

Remark 3.2. On the basis of the lemma mentioned in Remark 2.2, Chen and Yu [4] obtained the following oscillation criterion in the special case of the delay difference equation (1.2): If $0 < \alpha_0 \leq (\frac{k}{k+1})^{k+1}$, where

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-k}^{n-1} p(j),$$

then the condition

$$\limsup_{n\to\infty} \sum_{j=n-k}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0 - \alpha_0^2}\right) \tag{3.6}$$

implies that all solutions of (1.2) oscillate. In view of (3.6), it is interesting to ask if, provided that $0 < \alpha \leq -1 + \sqrt{2}$, the condition

$$\limsup_{n\to\infty} \sum_{j=\tau(n)}^{n} p(j) > 1 - \frac{1}{2}\left(1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}\right) \tag{3.7}$$

(which is weaker than (3.1)) is sufficient for all solutions of the delay difference equation (1.1) to be oscillatory. Nevertheless, it should be pointed out (cf. Remark 2.2) that, when $0 < \alpha \leq 6 - 4\sqrt{2}$ and (2.3) holds, the condition (3.2) in Theorem 3.1 is weaker than the above condition (3.7) and especially, when $\alpha = 6 - 4\sqrt{2} \simeq 0.3431457$, the lower bound in (3.7) is 0.8929094, while in (3.2) is 0.7573593.

We illustrate the significance of our results by the following example.

Example 3.1. Consider the equation

$$\Delta x(n) + p(n)x(n-2) = 0,$$

where

$$p(3n) = \frac{1474}{10000}, \quad p(3n+1) = \frac{1488}{10000}, \quad p(3n+2) = \frac{6715}{10000}, \qquad n = 0, 1, 2, \ldots.$$

Here $k = 2$ and it is easy to see that

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{1474}{10000} + \frac{1488}{10000} = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

and

$$\limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = \frac{1474}{10000} + \frac{1488}{10000} + \frac{6715}{10000} = 0.9677.$$

Observe that

$$0.9677 > 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0}\right) \simeq 0.967317794,$$

that is, condition (3.1) of Theorem 3.1 is satisfied and therefore all solutions oscillate. Also, condition (3.6) is satisfied. Observe, however, that

$$0.9677 < 1,$$

$$\alpha_0 = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

$$0.9677 < 1 - \left(1 - \sqrt{1 - \alpha_0}\right)^2 \simeq 0.974055774,$$

and therefore none of the conditions (1.4), (1.5) and (1.12) is satisfied.

If, on the other hand, in the above equation

$$p(3n) = p(3n+1) = \frac{1481}{10000}, \quad p(3n+2) = \frac{6138}{10000}, \qquad n = 0, 1, 2, ...,$$

it is easy to see that

$$\alpha_0 = \liminf_{n\to\infty} \sum_{j=n-2}^{n-1} p(j) = \frac{1481}{10000} + \frac{1481}{10000} = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

and

$$\limsup_{n\to\infty} \sum_{j=n-2}^{n} p(j) = \frac{1481}{10000} + \frac{1481}{10000} + \frac{6138}{10000} = 0.91.$$

Furthermore, it is clear that

$$p(n) \geq \frac{\alpha_0}{2} \text{ for all large } n.$$

In this case

$$0.91 > 1 - \frac{1}{4}\left(2 - 3\alpha_0 - \sqrt{4 - 12\alpha_0 + \alpha_0^2}\right) \simeq 0.904724375,$$

that is, condition (3.2) of Theorem 3.1 is satisfied and therefore all solutions oscillate. Observe, however, that

$$0.91 < 1,$$

$$\alpha_0 = 0.2962 < \left(\frac{2}{3}\right)^3 \simeq 0.2962963,$$

$$0.91 < 1 - \left(1 - \sqrt{1 - \alpha_0}\right)^2 \simeq 0.974055774,$$

$$0.91 < 1 - \frac{1}{2}\left(1 - \alpha_0 - \sqrt{1 - 2\alpha_0 - \alpha_0^2}\right) \simeq 0.930883291,$$

and therefore none of the conditions (1.4), (1.5), (1.12) and (3.6) is satisfied.

REFERENCES

[1] R. P. Agarwal, M. Bohner, S. R. Grace and D. O'Regan, *Discrete Oscillation Theory,* Hindawi Publishing Corporation, New York, 2005.

[2] G. E. Chatzarakis, R. Koplatadze and I. P. Stavroulakis, Oscillation criteria of first order linear difference equations with delay argument, *Nonlinear Anal.,* (in press).

[3] G. E. Chatzarakis and I. P. Stavroulakis, Oscillations of first order linear delay difference equations, *Aust. J. Math. Anal. Appl.* **3** (2006), No. 1, Art. 14, 11pp.

[4] M.-P. Chen and J. S. Yu, Oscillation for delay difference equation with variable coefficients, *Proceedings of the First International Conference on Difference Equations* (Edited by S. N. Elaydi *et al.*), Gordon and Breach, 1994, pp. 105–114.

[5] S. S. Cheng and G. Zhang, "Virus" in several discrete oscillation theorems, *Appl. Math. Lett.* **13** (2000), 9–13.

[6] Y. Domshlak, Sturmian comparison method in oscillation study for discrete difference equations, I, *Differential Integral Equations* **7** (1994), 571–582.

[7] Y. Domshlak, What should be a discrete version of the Chanturia-Koplatadze Lemma?, *Funct. Differ. Equ.* **6** (1999), 299–304.

[8] L. H. Erbe and B. G. Zhang, Oscillation of discrete analogues of delay equations, *Differential Integral Equations* **2** (1989), 300–309.

[9] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations: With Applications,* Clarendon Press, Oxford, 1991.

[10] I.-G. E. Kordonis and Ch.G. Philos, Oscillation and nonoscillation in linear delay or advanced difference equations, *Math. Comput. Modelling* **27** (1998), 11–21.

[11] G. Ladas, Oscillations of equations with piecewise constant arguments, *Differential Equations and Applications, Proceedings of the International Conference on Theory and Applications of Differential Equations,* March 21–25, 1988, Ohio University Press, Athens, pp. 64–69.

[12] G. Ladas, Explicit conditions for the oscillation of difference equations, *J. Math. Anal. Appl.* **153** (1990), 276–287.

[13] G. Ladas, Ch. G. Philos and Y. G. Sficas, Sharp conditions for the oscillation of delay difference equations, *J. Appl. Math. Simulation* **2** (1989), 101–111.

[14] Z. Luo and J. Shen, New results for oscillation of delay difference equations, *Comput. Math. Appl.* **41** (2001), 553–561.

[15] Z. Luo and J. Shen, New oscillation criteria for delay difference equations, *J. Math. Anal. Appl.* **264** (2001), 85–95.

[16] Ch. G. Philos, On oscillations of some difference equations, *Funkcial. Ekvac.* **34** (1991), 157–172.

[17] Ch. G. Philos, Oscillations in a nonautonomous delay logistic difference equation, *Proc. Edinburgh Math. Soc.* **35** (1992), 121–131.

[18] Ch. G. Philos and I. K. Purnaras, Oscillations in linear difference equations with variable coefficients, *J. Appl. Math. Stochastic Anal.* **4** (1991), 241–258.

[19] Ch.G. Philos, I.K. Purnaras and I.P. Stavroulakis, Sufficient conditions for the oscillation of delay difference equations, *J. Difference Equ. Appl.* **10** (2004), 419–435.

[20] J. H. Shen and I. P. Stavroulakis, Oscillation criteria for delay difference equations, *Electron. J. Differential Equations* **2001** (2001), No. 10, 1–15.

[21] I. P. Stavroulakis, Oscillations of delay difference equations, *Comput. Math. Appl.* **29** (1995), 83–88.

[22] I. P. Stavroulakis, Oscillation criteria for first order delay difference equations, *Mediterr. J. Math.* **1** (2004), 231–240.

[23] X. H. Tang and J. S. Yu, Oscillation of delay difference equation, *Comput. Math. Appl.* **37** (1999), 11–20.

[24] X. H. Tang and J. S. Yu, A further result on the oscillation of delay difference equations, *Comput. Math. Appl.* **38** (1999), 229–237.

[25] X. H. Tang and J. S. Yu, Oscillation of delay difference equations, *Hokkaido Math. J.* **29** (2000), 213–228.

[26] X. H. Tang and R. Y. Zhang, New oscillation criteria for delay difference equations, *Comput. Math. Appl.* **42** (2001), 1319–1330.

[27] J. S. Yu, B. G. Zhang and Z. C. Wang, Oscillation of delay difference equations, *Appl. Anal.* **53** (1994), 117–124.

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